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# Some Aspects of Abelian Categories

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***Abstract:*** **Here we discuss how zero objects and zero morphisms behave in an abelian category. We also provide proofs of isomorphism between the kernels and between the cokernels of a morphism. We discuss some properties of abelian category.**

***Keywords:*** **Zero object, Zero Morphism Kernel, Cokernel, Image, Coimage, Parallel Morphism, Abelian category, Preadditive category Additive Category, Preabelian Category.**

## 1. INTRODUCTION

Here we discuss the behaviour of zero object and zero morphism in an abelian category. We also discuss how kernels of a morphism are isomorphic and that of cokernels of a morphism are also isomorphic. We have proved some properties of abelian categories. We have given proof of some properties of exact sequences. We also try to prove that in an abelian category every bimorphism is an isomorphism.

## 2. PRELIMINARIES

For notions of category theory we shall in general follow the notation and terminology of Popescu [6]. However, we do deviate somewhat.

For C a category and A, B objects of **C**, Mor(A, B) denotes the set of morphisms from A to B.

We will also follow Popescu [6] for the definition of Preadditive, Additive and Preabelian and abelian category.

For kernel and cokernel we follow MacLane[2].

We follow the definition terminology of zero object and zero morphism from Popescu[6] and Pareigis[11].

We shall use the definition of Balanced category from Mitchel [3] Monomorphism from Schubert[5] and epimorphism and isomorphism from Pareigis[11].

## 3. MAIN RESULTS

**1. PROPOSITION**: **There is a unique isomorphism between each pair of zero objects in a category**.

**Proof** : Let us suppose that Z band X are two zero objects.Then Mor (Z,X) contains exactly one morphism and Mor (X,Z) consists only one morphism.Therefore the composition morphism Z→X→Z is unique and it will be $1_Z$, as Mor(Z,Z) has only $1_Z$.

Similarly the compositon morphism X→Z→X is also unique and it is $1_X$.

Hence Z and X are isomorphic.

**2. PROPOSITION: Let C be a category with zero objects. Then for objects A,B ∈ ob C , the map 0(A,B) : A → B is independent of the choice of our zero object Z.**

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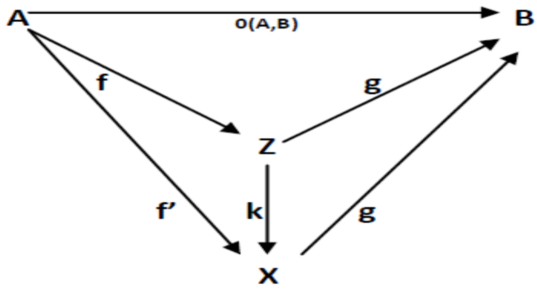
**Proof :** Let Z and X be zero objects in **C**. Then by definition of zero objects and by the definition of 0(A,B) : A → B we have the following diagram.....




Now gof = 0(A,B),where f,g are unique. Since X is zero object so f' and g' are also unique. Thus k is unique as both Z and X are zero objects. We need to show that 0(A,B) = g' o f'

By the uniqueness of g' and f' we have

g'ok = g and kof = f'.

now g'o(kof) = gof'

=> (g'ok)of =g'of'

=> gof = g'of'

=> 0(A,B) = g'of'

Thus 0(A,B) is independent of the choice of our zero object.

**Note : This proposition shows that 0(A,B) is unique for objects A and B.**

**Also in a category with zero objects , the zero morphism exists for any pair of objects (A,B) by its definition.**

***From now onwards we will denote zero morphism by just 0.***

**3. Composition with zero morphisms yields zero morphism.**

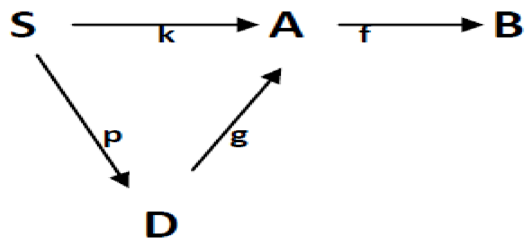
**Proof :** It is obvious from the definition of zero morphism.

**4 . Kernels are unique up to isomorphism.**

**Proof** : Let **C** be a category with zero object 0. Let f: A→B is a morphism and k: S →A and g : D →A are two kernels of f.

As g : D →A is kernel of f: A→B and fok = 0, therefore by definition of kernel there is a unique p: S → D such that gop = k…………………………………(i)

The following diagram shows—




Next , as k: S → A is a kernel of f: A→B and fog = 0,

then there exists a unique morphism h: D → S such that

koh = g …………………………………(ii)

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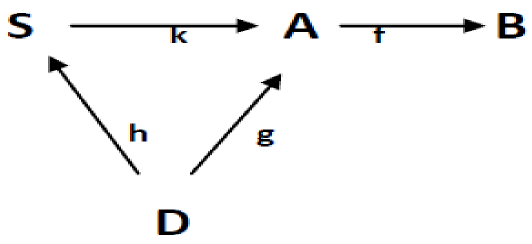



(i) = > (gop)oh = koh

= > go(poh) = g [from (ii)]

= > poh = $1_D$ [as kernel (g) is monomorphism ] ………(iii)

(ii) = > (koh)op = gop

= > ko(hop) = k [ from (i) ]

= > hop = $1_S$ [since kernel (k) is monomorphism]……….(iv)

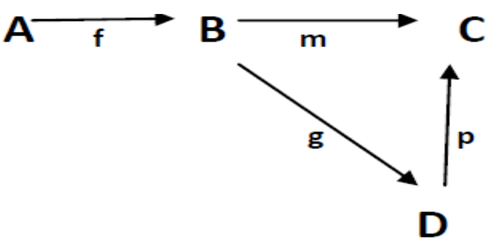
From (iii) and (iv) we have that both the kernels are isomorphic.

**5. Cokernels are unique up to isomorphism.**

**Proof** : Let **C** be a category with zero object 0. Let f: A→B be a morphism and m: B → C and g : B → D are cokernels of f: A→B

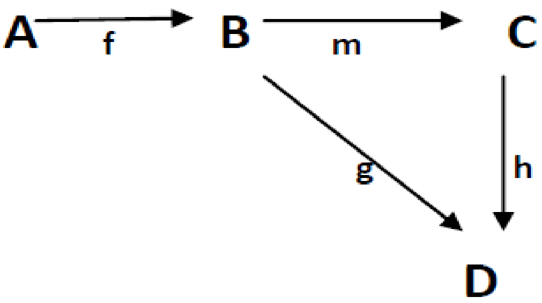
As g : B → D is a cokernel of f: A→B and mof = 0

Therefore by definition of cokernel we have a unique morphism p : D →C such that the following diagram........




commutes i.e. pog = m...............(i)

Also m: B → C is a cokernel of f: A→B and gof = 0, therefore by definition we have a unique morphism h : C →D such that the following diagram ......




commutes i.e. hom = g.........(ii)

(ii) = > po(hom) = pog

= > (poh)om = m [from (i)]

= > poh = $1_C$ …………………………………(iii)

(i) = > ho(pog) =hom

= > (hop)og = g [from (ii)]

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= > hop = $1_D$ ……………………………………(iv)

From (iii) and (iv) we have that the two cokernels are isomorphic.

***Note that kernels and cokernels need not exist for all morphisms.***

**6. In an abelian category C ,the following are equivalent-**

**(i) f: A→ B is monomorphism**

**(ii) 0 is kernel of f**

**Proof**:- Let f: A→ B be a monomorphism and g: C → A be its kernel.

So fog = 0.

= > g = 0 (since f is monomorphism)

= > 0 is kernel of f

Conversely suppose that 0: 0 → A is kernel of f: A→B.

Let g,h :C→A be such that fog =foh

=.> fo(g-h)=0 (since C is abelian category)

= > there exists a unique morphism k:C →0 such that 0ok = g-h

= > g-h = 0 .(since 0ok is zero morphism)

= >g = h

= > f is monomorphism.

**7. In an abelian category ,the following are equivalent-**

**(i) f: A→ B is epimorphism**

**(ii) 0 is cokernel of f**

**Proof**:- Let f: A→ B be a epimorphism and g:B→C be its cokernel.

So gof =0.

= > g=0 (since f is epimorphism)

= > 0 is cokernel of f

Conversely suppose that 0: B → 0 is cokernel of f: A→B.

Let g,h :B→C be such that gof=hof

=.> (g-h)of = 0 (since **C** is abelian category)

= > there exists a unique morphism k:0 →C such that ko0 =g-h

= > g-h = 0 .(since ko0 is zero morphism)

= > g=h

= > f is epimorphism.

**8. In an abelian category, if f:A→ B is monomorphism then ker(cokerf)= imf coincides with f:A→ B.**

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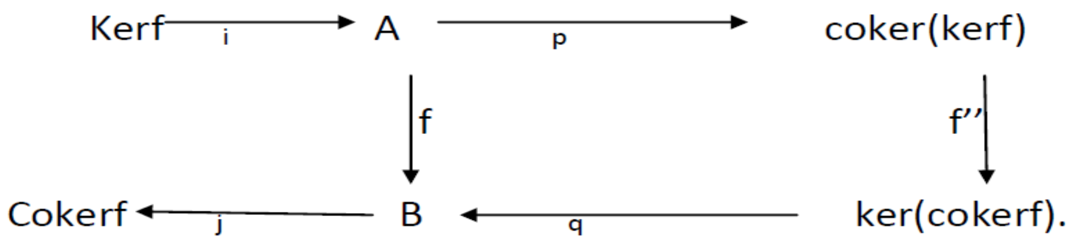
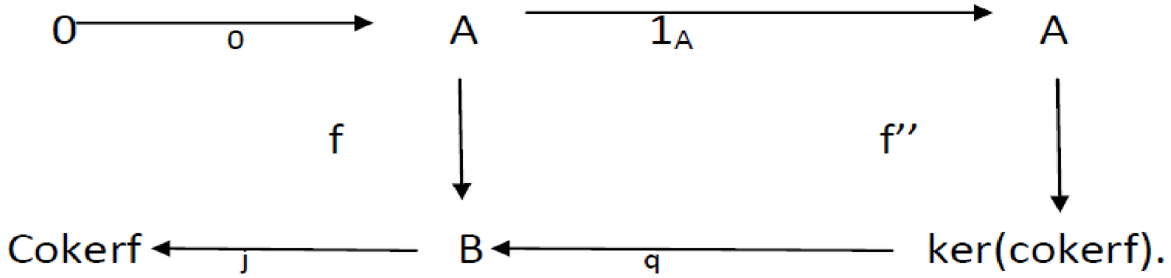
**Proof**:- Let us consider the diagram-




As f is monomorphism so kerf = 0 and coknel of 0: 0→A is the identity morphism $1_A$ : A→A. Thus the above diagram becomes

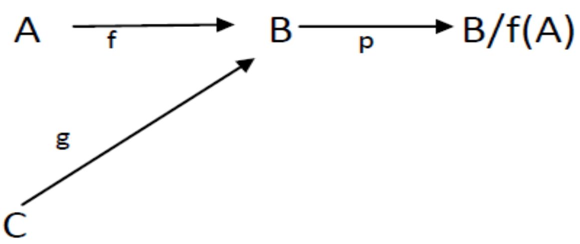



Since C is abelian category we have f'' is isomorphism. So A is isomorphic to ker(cokerf).

**For example**, in the category **Ab** of abelian groups, let f: A→B be a monomorphism. Then f(A)≌A.

Also p: B→ B/f(A is cokernel of f: A→B .

Consider the diagram



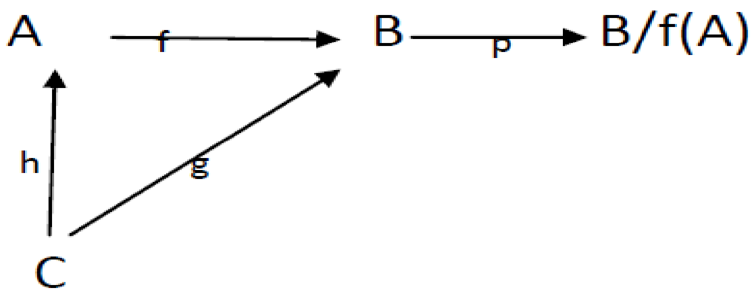

such that pog = 0.Let us define h: C→A such that h(c)= g(c) which is well defined.

Now foh(c) =f(h(c))

= h(c)

= g(c)

= > foh =g. Also his unique i.e. the diagram




commutes.

Hence kernel of p: B→ B/f(A) ≌ A is f: A→B .

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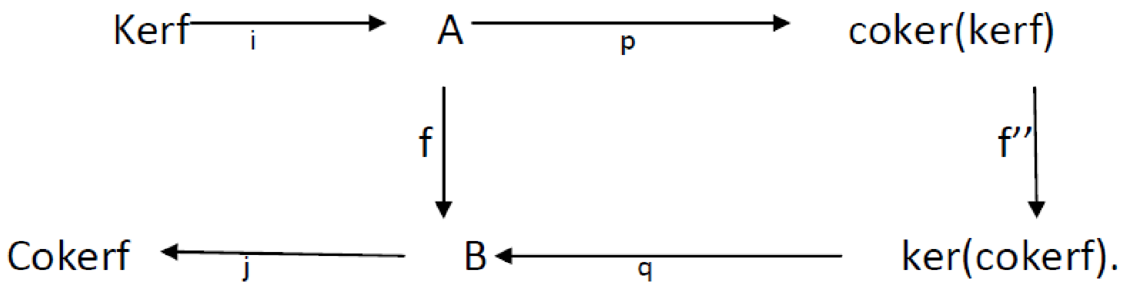
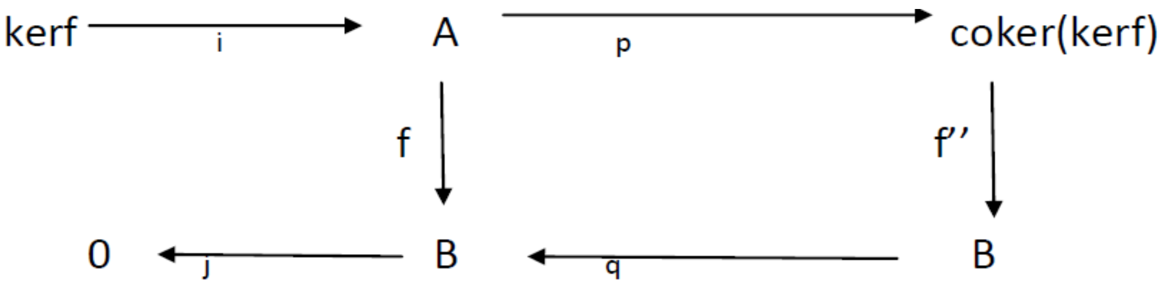
**9. In an abelian category, if f:A→ B is epimorphism then coker(kerf) coincides with f:A→ B.**

**Proof** : Let us consider the diagram-




As f is epimorphism so cokerf = 0 and kernel of 0: B→0 is the identity morphism $1_B$ : B→B. Thus the above diagram becomes




Since C is abelian category we have f'' is isomorphism. So B is isomorphic to coker(kerf).

**10. A morphism f : A → B is an epimorphism iff imf= B.**

**Proof**:- f is epimorphism iff coker(kerf)=B

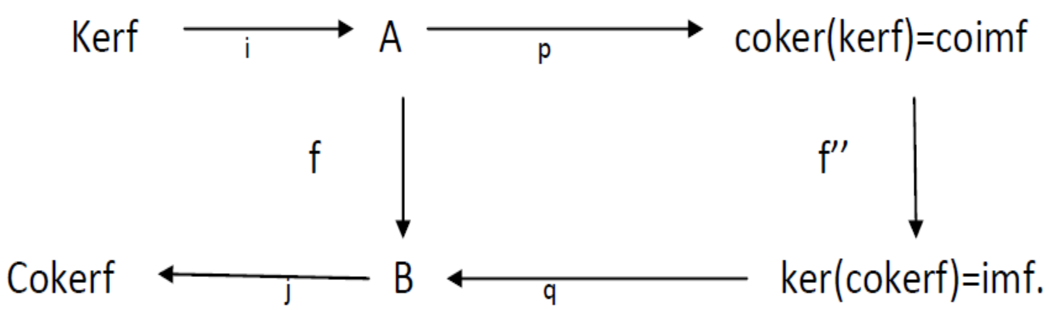
Since **C** is abelian so coimf ≅ imf i.e. coker(kerf) ≅ ker(cokerf)

= > B≅ker(cokerf)

= > B≅imf.

**11. Generally in any category C every isomorphism is bimorphism but the converse may not be true.But in case of an abelian category every bimorphism is isomorphism.**

**Proof**: Let **C** be an abelian category and f : A → B be a bimorphism i.e. f is monomorphism as well as epimorphism. Let us consider the diagram




Now f is monomorphism = > 0 is kernel of f

And cokernel of 0→A is the identity morphism $1_A$: A→A.

Also fis epimorphism = > 0 is cokernel of f

And kernel of B→0 is the identity morphism $1_B$ :B→B.

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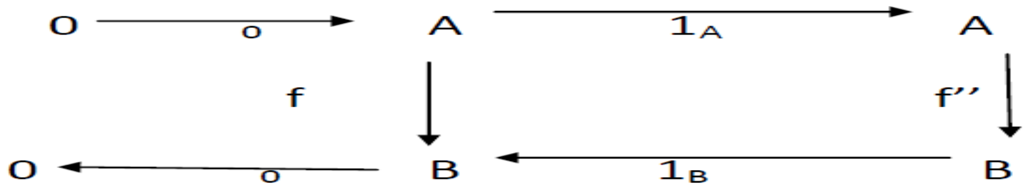
Thus the above diagram becomes




Which gives $f = 1_B \text{ o } f'' \text{ o } 1_A$

$= > f = f''$

$= >$ f is isomorphism as f'' is isomorphism.

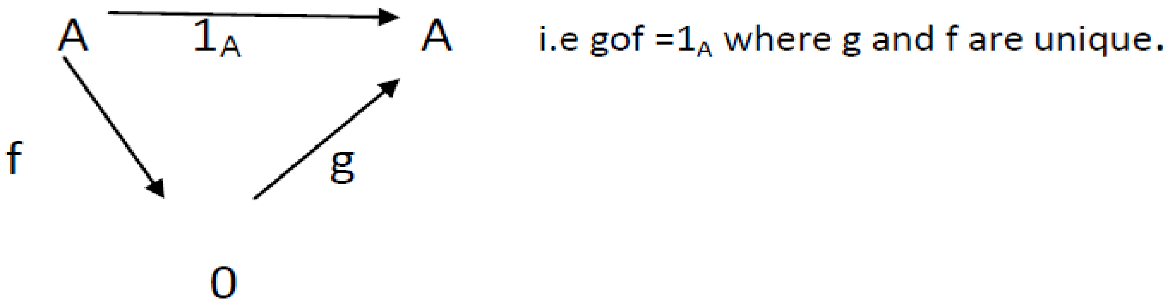
**12. Let the category C has a zero object 0.**

**(a) A ∈ obC is a zero object iff $1_A$ is zero morphism.**

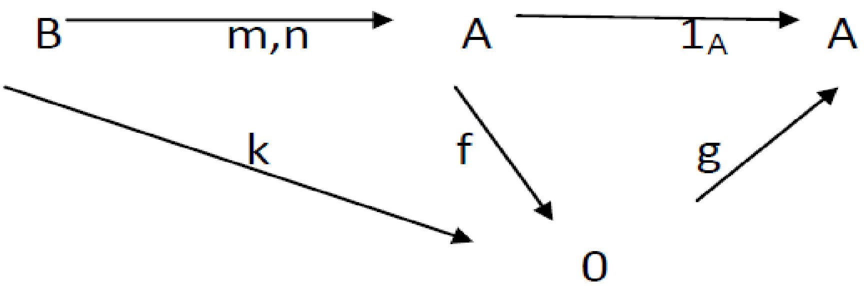
**(b) if m : A →0 is a monomorphism, then A is a zero object.**

**Proof:** (a) Let A be a zero object. Then $1_A$ is the only morphism from A→ A as A is zero object. And hence $1_A$ is zero morphism.

Conversely suppose that $1_A$ is zero morphism. Then the following diagram commutes




Let m,n : B → A. Then $1_A$og = g and $1_A$ o f = f. As 0 is zero object we have a unique morphism k: B →0 such that from the diagram




We have fom= k = fon.

Now gofom = gofon

$= > 1_A$ om $= 1_A$ on

$= >$ m = n.

Thus A is terminal object.

Similarly it can be shown that A is initial object .

Hence A is zero object.

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(b) Let m: A→0 be monomorphism

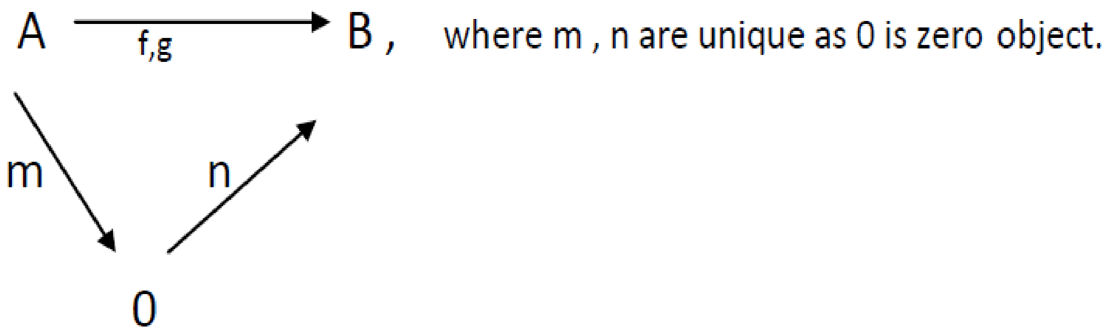
Let us consider $B \xrightarrow{f,g} A \xrightarrow{m} 0$ such that

mof = mog

= > f = g [ as m is monomorphism]

Thus A is terminal object.

Let f,g : A→B be two morphisms. Then we have the following diagram




Such that nom = f, nom = g

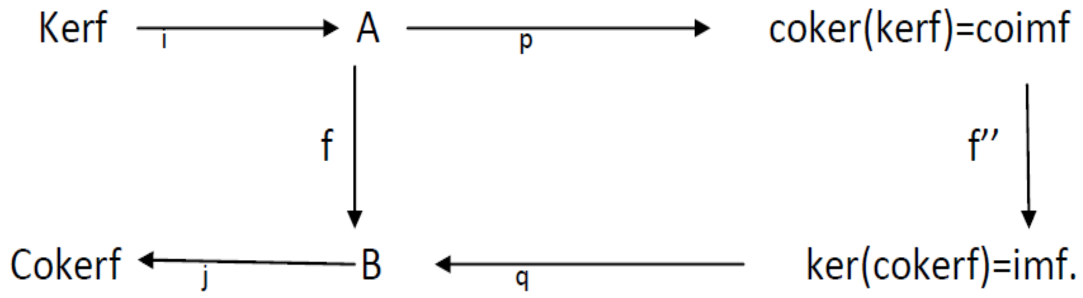
= > f = g

Therefore A is initial object.

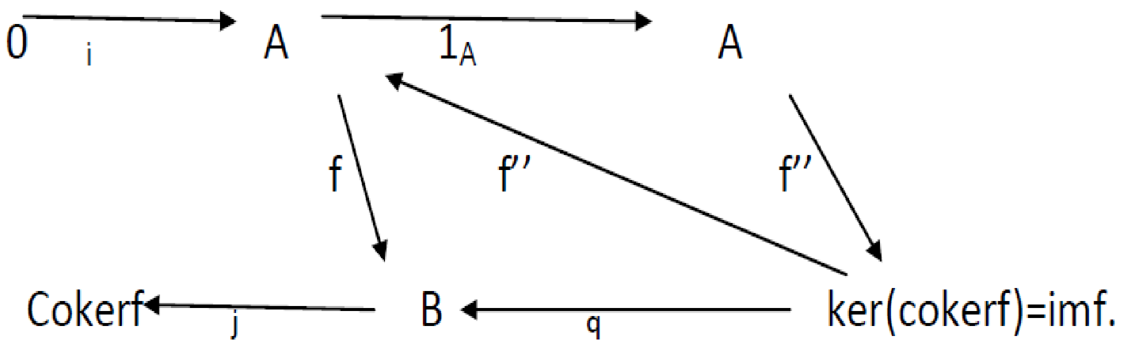
Hence A is zero object.

**13. If f : A →B is monomorphism in an abelian category then it is kernel of some other morphism.**

**Proof:** Let us consider the diagram




As f is monomorphism so kerf =0 , coker(0) = A and p = $1_A$So the above diagram becomes




Also C is abelian so f'' is isomorphism ,by definition.

Therefore fof'' = q. Which shows that f : A→ B is kernel of j: B →cokerf.

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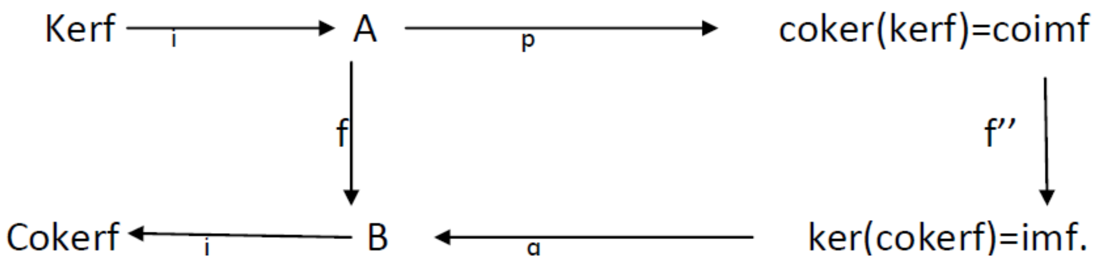
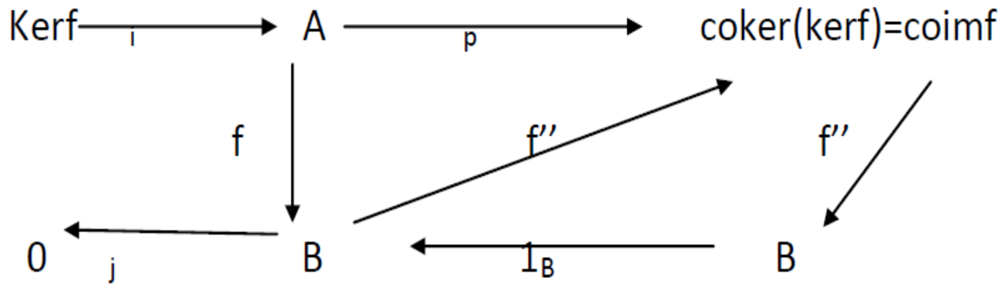
**14. Any epimorphism in an abelian category is the cokernel of a morphism.**

**Proof** : Let us consider the diagram




As f is epimorphism so cokerf is 0. Therefore ker(0)= B

And q = $1_B$. Now the above diagram becomes




As C is abelian category so f'' is isomorphism, by definition. Thus f'' o f = p.Which shows that f: A →B is cokernel of i : kerf →A.

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